TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 323, Number 2, February 1991

AN L^2-COHOMOLOGY CONSTRUCTION OF UNITARY HIGHEST WEIGHT MODULES FOR $\mathrm{U}(p, q)$

LISA A. MANTINI

ABSTRACT. In this paper a geometric construction is given of all unitary highest weight modules of $G = \mathrm{U}(p, q)$. The construction is based on the unitary model of the kth tensor power of the metaplectic representation in a Bargmann-Segal-Fock space of square-integrable differential forms. The representations are constructed as holomorphic sections of certain vector bundles over G/K, and the construction is implemented via an integral transform analogous to the Penrose transform of mathematical physics.

0. INTRODUCTION

A major area of research in the representation theory of Lie groups is the explicit analysis of naturally occurring unitary representations. One aspect of this problem is that of producing geometric constructions of the unitary representations of a semisimple Lie group G. Here we consider unitary highest weight representations of $G = \mathrm{U}(p, q)$. These representations have been classified algebraically in various ways (see [EHW, J1, A]), and have been constructed as spaces of vector-valued functions on the generalized unit disk [D, DS]. Most singular highest weight representations appear in the L^2-cohomology construction of [RSW, Z].

Our approach starts with the kth tensor power of the metaplectic representation of $\mathrm{U}(p, q)$ in the L^2-cohomology space $\mathscr{H}^{0,qk}(\mathbb{C}^{n\times k})$, as defined by Blattner and Rawnsley [BR]. The reductive dual pair $(\mathrm{U}(p, q), \mathrm{U}(k))$ acts on this space. Results of Kashiwara and Vergne [KV] decompose this space into a sum of irreducible representations $\{\sigma_\lambda\}$ indexed by a set $\Lambda \subset \widehat{\mathrm{U}(k)}$ of representations of $\mathrm{U}(k)$. All unitary highest weight modules of $\mathrm{U}(p, q)$ appear in this way for some k [J2, EP]. Each irreducible $G = \mathrm{U}(p, q)$-module corresponds to a vector bundle over G/K, as follows. Corresponding to a particular representation σ_λ of G, each element ζ of G/K determines a subspace $V_{s(\lambda)}(\zeta)$

Received by the editors October 31, 1989. Results presented in the Special Session on Geometric Methods in Representation Theory at the American Mathematical Society Meeting in Los Angeles, California, November 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E45.

Key words and phrases. Semisimple Lie groups, unitary representations, highest weight modules, metaplectic representation, integral transform.

Research partially supported by NSF Grants DMS-8641450 and RII-8610676 (EPSCoR).

of $\mathbb{C}^{n\times k}$, and the vector bundle $\mathbf{E}_\lambda$ over G/K is defined so that the fiber over ζ is essentially the L^2-cohomology space $\mathscr{H}_\lambda^{0,qs}(V_s(\zeta))$. The construction now proceeds by an integral transform which is given by restriction of a differential form to the subspace $V_s(\zeta)$ followed by projection onto an L^2-cohomology class. This produces a holomorphic section of the vector bundle $\mathbf{E}_\lambda$, and the transform is injective.

The integral transform used here is modelled after the classical Radon transform (see [H]) and the Penrose transform of mathematical physics (see [We]). Other authors have used integral-geometric techniques to study explicit realizations of unitary representations (see [PR, E, EPt, EPW, Du]).

This result generalizes the author's previous construction of the ladder representations of $\mathrm{U}(p, q)$ (see [M1, Theorem 3.1; M2, Theorem 4.8]), which occur here in the case $k = 1$. In the earlier works, the subspaces $V_s(\zeta)$ were either positive p-planes ($s = 0$) or negative q-planes ($s = 1$) in $\mathbb{C}^{p,q}$. Here the spaces $V_s(\zeta)$ are direct sums of $k - s$ positive p-planes and s negative q-planes. The methods of the earlier works then generalize very easily to the present case.

In §1 we summarize the Blattner-Rawnsley construction [BR] of the metaplectic representation of $\mathrm{U}(p, q)$ in the L^2-cohomology space $\mathscr{H}^{0,qk}(\mathbb{C}^{n\times k})$. In §2 we describe the Kashiwara-Vergne decomposition [KV] of this representation. In §3 we construct the vector bundles over G/K. Finally, in §4 we construct the integral transform.

The author gratefully acknowledges financial support from NSF Grant DMS–8641450 at the University of Utah under H. Rossi and support during the 1986–87 academic year from the Sonderforschungsbereich 170 at the Universität Göttingen. M. Davidson was very helpful in explaining the results of Kashiwara and Vergne. Several other friends and colleagues provided help and support while this work was being completed, notably D. Garfinkle, J. Johnson, G. Olafsson, R. Stanke, H. Thorleifsson, and B. Ørsted.

1. The oscillator representation in L^2-cohomology

In this section we describe a model of the metaplectic representation of $\mathrm{U}(p, q)$ in an L^2-cohomology space of square-integrable differential forms. Our definition of L^2-cohomology follows the conventions of Blattner and Rawnsley [BR]. A theorem of Blattner and Rawnsley [BR] describes the unitary structure of the $\mathrm{U}(p, q)$ action on these L^2-cohomology spaces. The following section will describe the decomposition of these spaces into irreducible subspaces.

1.1. **Definition.** By $\mathbb{C}^{r\times s}$ we mean the space of $r \times s$ complex matrices. We define a positive definite hermitian form on $\mathbb{C}^{r\times s}$ by $\langle z, w\rangle := \mathrm{tr}({}^*w z)$, for $z, w \in \mathbb{C}^{r\times s}$. We define $|z|^2 := \langle z, z\rangle$. By $dm(z)$ we mean Lebesgue measure on $\mathbb{C}^{r\times s}$ normalized so that

$$\int_{\mathbb{C}^{r\times s}} e^{-|z|^2}\, dm(z) = 1.$$

We also list here several integration formulas which will be needed frequently.

1.2. **Lemma.** (1) *If* $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, *then*

$$\int_{\mathbb{R}} e^{-\alpha t^2+2\beta t} dt = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left(\frac{\beta^2}{\alpha}\right).$$

(2) *If* $j, k \in \mathbb{N}$, *and if* $\alpha \in \mathbb{C}$ *has* $\Re(\alpha) > 0$, *then*

$$\int_{\mathbb{C}} z^k \bar{z}^j e^{-\alpha|z|^2}\, dm(z) = \begin{cases} 0, & j \neq k, \\ k!\,\alpha^{-(k+1)}, & j = k. \end{cases}$$

(3) *If* $U \in \mathbb{C}^{r\times r}$ *satisfies* $(U + {}^*U) \gg 0$, *and if* φ *is a holomorphic (or antiholomorphic) function of* $z \in \mathbb{C}^{r\times s}$ *for which* $\int \varphi(z)\exp(-\langle Uz, z\rangle)\, dm(z)$ *converges absolutely, then*

$$\int_{\mathbb{C}^{r\times s}} \varphi(z) e^{-\langle Uz, z\rangle}\, dm(z) = \frac{\varphi(0)}{\det U^s}.$$

1.3. **Definition.** We now fix positive integers p and q, and we let $n = p+q$. By $\mathbb{C}^{p,q}$ we mean $\mathbb{C}^{p+q}$ endowed with a fixed hermitian form of signature (p, q). By $G = \mathrm{U}(p, q)$ we mean the group of all invertible linear transformations of $\mathbb{C}^{p,q}$ preserving this form. In particular, fixing coordinates on $\mathbb{C}^{p,q}$ so that the matrix of the form is given by

$$I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

then

$$\mathrm{U}(p, q) = \{ g \in \mathrm{GL}(n, \mathbb{C}) \mid {}^*g I_{p,q} g = I_{p,q} \},$$

so that $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{U}(p, q)$ (where A is $p \times p$, etc.) if and only if

$$(1.4) \qquad {}^*AA - {}^*CC = I_p, \qquad {}^*AB = {}^*CD, \qquad \text{and} \qquad {}^*DD - {}^*BB = I_q.$$

1.5. **Definition.** Fix a positive integer k. We wish to consider the reductive dual pair $(\mathrm{U}(p, q), \mathrm{U}(k))$ inside $\mathrm{Sp}(nk, \mathbb{R})$. We let $\mathbf{W} := \mathbb{C}^{n\times k} = \mathbb{C}^{p,q} \otimes \mathbb{C}^k$. We fix a hermitian form on $\mathbf{W}$ which is given by

$$\langle\langle z, w\rangle\rangle := \langle I_{p,q} z, w\rangle = \mathrm{tr}({}^*w I_{p,q} z),$$

for $z, w \in \mathbf{W}$. This form has signature (pk, qk) on $\mathbf{W}$, and is invariant under the left action of $\mathrm{U}(p, q)$ and the right action of $\mathrm{U}(k)$ on $\mathbf{W}$. The imaginary part of $\langle\langle \cdot, \cdot\rangle\rangle$ gives a symplectic form on the $2nk$-dimensional real vector space underlying $\mathbf{W}$, thus we have realized $\mathrm{U}(p, q)$ and $\mathrm{U}(k)$ as subgroups of $\mathrm{Sp}(nk, \mathbb{R})$ which clearly centralize each other.

We now wish to construct L^2-cohomology spaces for $\mathbf{W}$. We thus must define an inner product structure on the space $\mathscr{E}^{r,s}(\mathbf{W})$ of smooth differential (r, s)-forms on $\mathbf{W}$.

1.6. **Definition.** For $z \in \mathbf{W}$, we write

$$z = \begin{pmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nk} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where z_1 is $p \times k$ and z_2 is $q \times k$. We then note that $\langle\langle z, w \rangle\rangle = \langle z_1, w_1 \rangle - \langle z_2, w_2 \rangle$. Now let I, J, K, and L be multi-indices for our variables $z_{11}, \ldots, z_{nk}$, i.e.,

$$I, J, K, L \subseteq \{(1, 1), \ldots, (1, k), \ldots, (n, 1), \ldots, (n, k)\}$$

with I, K having cardinality r and J, L having cardinality s. We then define

$$\langle \varphi dz_I \wedge d\bar{z}_J \mid \psi dz_K \wedge d\bar{z}_L \rangle := \delta_{IK}\delta_{JL} \int_{\mathbf{W}} \varphi(z)\overline{\psi(z)}\, dm(z).$$

This inner product converges on $\mathscr{E}_c^{r,s}(\mathbf{W})$. We let $\mathscr{L}_2^{r,s}(\mathbf{W})$ denote the completion of $\mathscr{E}_c^{r,s}(\mathbf{W})$ with respect to $\langle \cdot | \cdot \rangle$. The Hilbert space structure of $\mathscr{L}_2^{r,s}(\mathbf{W})$ depends on the arbitrary choice of hermitian form $\langle \cdot | \cdot \rangle$ and thus is not $\mathrm{U}(p, q)$-invariant. As a topological vector space, however, $\mathscr{L}_2^{r,s}(\mathbf{W})$ is independent of the choice of hermitian form.

The coboundary operator in cohomology comes from the covariant differential of a line bundle over $\mathbf{W}$ via geometric quantization, as in [BR].

1.7. **Definition.** For $\omega \in \mathscr{E}^{r,s}(\mathbf{W})$, define $\nabla\omega$ by

$$\nabla\omega := e^{-\frac{1}{2}\langle\langle z, z \rangle\rangle}\, \overline{\partial}(\omega e^{\frac{1}{2}\langle\langle z, z \rangle\rangle}),$$

that is, if $\omega = \varphi dz_I \wedge d\bar{z}_J$, then

$$\nabla\omega = \sum_{i=1}^{n}\sum_{j=1}^{k} \left(\frac{\partial\varphi}{\partial\bar{z}_{ij}} + \frac{1}{2}\varepsilon_i z_{ij}\varphi \right) d\bar{z}_{ij} \wedge dz_I \wedge d\bar{z}_J$$

where

$$\varepsilon_i = \begin{cases} +1, & 1 \le i \le p, \\ -1, & p+1 \le i \le n. \end{cases}$$

The domain of ∇ includes $\mathscr{E}_c^{r,s}(\mathbf{W})$, so is dense in $\mathscr{L}_2^{r,s}(\mathbf{W})$, and ∇ has a closure in $\mathscr{L}_2^{0,s}(\mathbf{W})$ [BR]. We extend ∇ to be a closed operator, still denoted by ∇; hence $\ker\nabla$ is closed.

1.8. **Definition.** We define the L^2-cohomology space of harmonic $(0, s)$-forms on $\mathbf{W}$ by

$$\mathscr{H}^{0,s}(\mathbf{W}) := \ker\nabla \cap \ker\nabla^* \cap \mathscr{L}_2^{0,s}(\mathbf{W}),$$

where ∇^* denotes the adjoint of ∇ with respect to $\langle \cdot | \cdot \rangle$.

Note that $\mathscr{H}^{0,s}(\mathbf{W})$ depends on the choice of hermitian form $\langle \cdot | \cdot \rangle$.

1.9. **Theorem** [BR]. *If $s \neq qk$, then $\mathscr{H}^{0,s}(\mathbf{W}) = \{0\}$, and $\mathscr{H}^{0,qk}(\mathbf{W})$ is given by*

$$\mathscr{H}^{0,qk}(\mathbf{W}) = \Big\{ \varphi d\bar{z}_2 \mid \varphi(z) = f(z)e^{-\frac{1}{2}|z|^2} \text{ where } f \text{ is holomorphic in } z_1, \bar{z}_2 \text{ and } \int_{\mathbf{W}} |f(z)|^2 e^{-|z|^2}\, dm(z) < \infty \Big\}.$$

This is a Bargmann-Segal-Fock space in which the set of all such differential forms ω with f a polynomial in $z_1, \bar{z}_2$ is dense, and so $\mathscr{H}^{0,qk}(\mathbf{W})$ has a reproducing kernel.

1.10. **Lemma** [B]. *The orthogonal projection $P\colon \mathscr{L}_2^{0,qk}(\mathbf{W}) \to \mathscr{H}^{0,qk}(\mathbf{W})$ is given by $P(\varphi d\bar{z}_I) = 0$ unless $z_I = z_2$, and*

$$P(\varphi d\bar{z}_2)(z) = d\bar{z}_2 \int_{\mathbf{W}} \varphi(w) K(z, w)\, dm(w),$$

where $K(z, w) = e^{-\frac{1}{2}|z|^2} e^{\langle z_1, w_1\rangle + \langle w_2, z_2\rangle} e^{-\frac{1}{2}|w|^2}$.

It now remains to describe the $\mathrm{U}(p, q)$-action on $\mathscr{H}^{0,qk}(\mathbf{W})$.

1.11. **Definition.** For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p, q)$, define $\sigma(g)$ on $\omega = \varphi d\bar{z}_2 \in \mathscr{H}^{0,qk}(\mathbf{W})$ by $\sigma(g) := P \circ l(g)$, i.e.,

$$\begin{aligned}(\sigma(g)\omega)(z) &= P(\varphi(g^{-1}z) d\overline{(g^{-1}z)_2}) \\ &= \det D^k\, d\bar{z}_2 \int_{\mathbf{W}} \varphi(g^{-1}w) K(z, w)\, dm(w)\end{aligned}$$

where K is given in 1.10.

1.12. **Theorem** [BR]. *The mapping σ gives a well-defined, continuous representation of G on $\mathscr{H}^{0,qk}(\mathbf{W})$ which is unitary with respect to $\langle\cdot|\cdot\rangle$.*

This representation σ is often referred to as the kth tensor power of the oscillator representation of $\mathrm{U}(p, q)$. Up to a character, σ is the restriction to $\mathrm{U}(p, q)$ of the metaplectic representation of the double cover $\mathrm{Mp}(nk, \mathbb{R})$ of $\mathrm{Sp}(nk, \mathbb{R})$. A construction of the metaplectic representation on a Fock space model is given as follows [D].

1.13. **Definition.** Let $\mathscr{F}$ denote the Fock space

$$\mathscr{F} := \Big\{ f : \mathbf{W} \to \mathbb{C} \mid f \text{ is holomorphic in } z_1, \bar{z}_2 \text{ and } \int_{\mathbf{W}} |f(z)|^2 e^{-|z|^2}\, dm(z) < \infty \Big\}.$$

Let $\mathbf{H}$ denote the Heisenberg group $\mathbf{H} := \mathbf{W} \ltimes \mathbb{R}$ with multiplication law $(z, t)(w, s) := (z + w, t + s + \Im\langle\langle z, w\rangle\rangle)$.

The group $\mathrm{Sp}(nk, \mathbb{R})$ acts as a group of automorphisms of $\mathbf{H}$ which preserve the center. Let τ be an irreducible unitary representation of $\mathbf{H}$ on $\mathscr{F}$ for which

$\tau(0, t) = e^{-it}$. The Stone-von Neumann Theorem implies that any irreducible unitary representation of $\mathbf{H}$ is uniquely determined by its restriction to the center, whenever this restriction is nontrivial. Thus, for any $g \in \mathrm{Sp}(nk, \mathbb{R})$, $\tilde{\tau}(z, t) := \tau(gz, t)$ gives another unitary irreducible representation of $\mathbf{H}$ on $\mathcal{F}$ which agrees with τ on the center of $\mathbf{H}$, hence these must be equivalent by some operator $\mu(g)$ satisfying

$$\mu(g)\tau(z, t) = \tau(gz, t)\mu(g).$$

These operators $\mu(g)$ determine a representation of $\mathrm{Mp}(nk, \mathbb{R})$, not $\mathrm{Sp}(nk, \mathbb{R})$, on $\mathcal{F}$. By suitably choosing τ, Davidson [D] constructs $\mu(g)$ for $g \in \mathrm{U}(p, q)$ as follows.

1.14. **Theorem** [D, Theorem 1.12]. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p, q)$ *and let* $f \in \mathcal{F}$. *The operator* $\mu(g)$ *is given by*

$$(\mu(g)f)(z) = \frac{e^{\langle CA^{-1}z_1, z_2\rangle}}{\det A^k} \int_{\mathbf{W}} f(w) e^{\langle A^{-1}z_1, w_1\rangle + \langle w_2, D^{-1}z_2\rangle} \times e^{-\langle A^{-1}Bw_2, w_1\rangle} e^{-|w|^2}\, dm(w)$$

and defines a continuous unitary representation of $\mathrm{U}(p, q)$ *on* $\mathcal{F}$.

1.15. **Theorem.** *If* $\Psi : \mathcal{F} \to \mathcal{H}^{0,qk}(\mathbf{W})$ *is defined by* $(\Psi f)(z) = f(z)e^{-\frac{1}{2}|z|^2} d\bar{z}_2$, *then for all* $g \in \mathrm{U}(p, q)$,

$$\sigma(g) \circ \Psi = \det g^k\, \Psi \circ \mu(g).$$

Proof. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p, q)$ and $K(z, w)$ as in Lemma 1.10, we use Lemma 1.2(3) to verify the equality

$$\int_{\mathbf{W}} K(z, gw)K(w, u)\, dm(w) = \frac{1}{|\det A|^{2k}} e^{-\frac{1}{2}(|z|^2 + |u|^2)} e^{\langle A^{-1}z_1, u_1\rangle + \langle u_2, D^{-1}z_2\rangle} e^{\langle CA^{-1}z_1, z_2\rangle - \langle A^{-1}Bu_2, u_1\rangle}.$$

Since $\det g = \det D \det{}^* A^{-1}$, the theorem then follows directly from Lemma 1.10. □

It remains to decompose $\mathcal{H}^{0,qk}(\mathbf{W})$ into irreducible components.

2. Decomposition of the oscillator representation

We wish to decompose the action of σ on $\mathcal{H}^{0,qk}(\mathbf{W})$ into irreducible components. We make use of the duality of representations of G with representations of $\mathrm{U}(k)$.

2.1. **Definition.** If $a \in \mathrm{U}(k)$, we define $\rho(a)$ on $\omega = \varphi d\bar{z}_2 \in \mathcal{H}^{0,qk}(\mathbf{W})$ by

$$(\rho(a)\omega)(z) := \omega(za) = \det \bar{a}^q \varphi(za) d\bar{z}_2.$$

Clearly the actions $\sigma(g)$ and $\rho(a)$ commute, for $g \in G$ and $a \in \mathrm{U}(k)$. For $\lambda \in \widehat{\mathrm{U}(k)}$, we let $\pi_\lambda = d_\lambda \int_{\mathrm{U}(k)} \rho(a)\overline{\chi_\lambda(a)}\,da$ be the projection onto the λ-isotypic component in $\mathscr{H}^{0,qk}(\mathbf{W})$. Let

$$\Lambda := \{ \lambda \in \widehat{\mathrm{U}(k)} \mid \pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W})) \neq \{0\} \}.$$

Of course, by the Peter-Weyl Theorem we know that

$$\mathscr{H}^{0,qk}(\mathbf{W}) = \bigoplus_{\lambda\in\Lambda} \pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W})).$$

Results of Kashiwara and Vergne (see [KV or D]) show that $\pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$ is isotypic for $\mathrm{U}(p, q)$ and describe the elements of $\pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$ in terms of "pluriharmonic" polynomials. Here we wish to find a distinguished irreducible subspace of $\pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$. Let

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_k} \end{pmatrix} =: e^{i(\theta_1, \dots, \theta_k)} \mid \theta_j \in \mathbb{R} \right\}$$

be a maximal torus in $\mathrm{U}(k)$.

2.2. **Definition.** If $m \in \mathbb{Z}^k$, define

$$\mathscr{H}_m^{0,qk}(\mathbf{W}) := \{ \omega \in \mathscr{H}^{0,qk}(\mathbf{W}) \mid \omega(z) = f(z)e^{-\frac{1}{2}|z|^2} d\bar{z}_2 \text{ where } f(ze^{i(\theta_1, \dots, \theta_k)}) = e^{i(m_1\theta_1 + \cdots + m_k\theta_k)} f(z) \}.$$

We think of f as having homogeneity degree m_i in the ith column of z. Note that $\mathscr{H}_m^{0,qk}(\mathbf{W})$ is $\mathrm{U}(p, q)$-invariant.

For future use in picking out a distinguished irreducible component in $\pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$, we construct here an example of a vector in $\mathscr{H}_m^{0,qk}(\mathbf{W})$.

2.3. Example. Let $m \in \mathbb{Z}^k$ be a nonincreasing sequence of integers with $r = r(m)$ strictly positive entries and $s = s(m)$ strictly negative entries, so that there are $k - (r + s)$ zeros in the middle. If $r \leq p$ and $s \leq q$, we define $z(i)$, for $1 \leq i \leq r$, to be the lower left $i \times i$ submatrix of z_1 and $z[j]$, for $1 \leq j \leq s$, to be the upper right $j \times j$ submatrix of z_2, so that

$$z = \left(\begin{array}{ccc} * & * & * \\ z(i) & * & * \\ \hline * & * & z[j] \\ * & * & * \end{array} \right).$$

We then define

$$f_m(z) := \det z(1)^{m_1 - m_2} \cdots \det z(r-1)^{m_{r-1} - m_r} \det z(r)^{m_r} \det \overline{z[1]}^{\,m_{k-1} - m_k} \cdots \det \overline{z[s-1]}^{\,m_{k-(s-1)} - m_{k-(s-2)}} \det \overline{z[s]}^{\,(-m_{k-(s-1)})}$$

and we define

$$\omega_m(z) := f_m(z)e^{-\frac{1}{2}|z|^2}d\bar{z}_2. \tag{2.4}$$

It follows from Definition 2.2 and Theorem 1.5 that

$$\omega_m \in \mathscr{H}_m^{0,qk}(\mathbf{W}). \tag{2.5}$$

2.6. **Theorem** [KV]. *Let* $\lambda \in \widehat{\mathrm{U}(k)}$, *and identify* λ *with its highest weight, so that* $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k$ *is a nonincreasing sequence of integers. Then* $\omega_\lambda \in \pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$ *and* ω_λ *is the highest weight vector with respect to the upper triangular Borel subgroup for the action* $l \times \rho$ *of* $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$. *The weight of* ω_λ *under* l *is*

$$(0, \dots, 0, -\lambda_r, \dots, -\lambda_1) \oplus (-\lambda_k, \dots, -\lambda_{k-s+1}, 0, \dots, 0) \in \mathbb{Z}^p \oplus \mathbb{Z}^q.$$

Finally, for any $\lambda \in \mathbb{Z}^k$, $\lambda \in \Lambda$ *if and only if*

$$\lambda_1 \geq \cdots \geq \lambda_k, \qquad r(\lambda) \leq p, \quad \text{and} \quad s(\lambda) \leq q. \tag{2.7}$$

2.8. **Definition.** Denote by $\mathscr{H}(\lambda)$ the irreducible subspace of $\pi_\lambda(\mathscr{H}^{0,qk}(\mathbf{W}))$ containing ω_λ. It follows from (2.5) and Theorem 2.6 that

$$\mathscr{H}(\lambda) \subseteq \mathscr{H}_\lambda^{0,qk}(\mathbf{W}) \tag{2.9}$$

and that

$$\mathscr{H}^{0,qk}(\mathbf{W}) = \bigoplus_{\lambda\in\Lambda} d_\lambda \mathscr{H}(\lambda).$$

2.10. **Theorem** [J2]. *All unitary highest weight modules of* $\mathrm{U}(p, q)$ *are obtained in this way as modules* $\mathscr{H}(\lambda)$ *where* $\lambda \in \Lambda \subseteq \mathbb{Z}^k$, *i.e.,* λ *satisfies condition* (2.7) *of Theorem* 2.6.

That all unitary highest weight modules of $\mathrm{U}(p, q)$ and $\mathrm{Mp}(n, \mathbb{R})$ are obtained in this way was conjectured in [KV] and proven in [EP]. This also follows from the more general result of [EHW] classifying all unitary highest weight modules.

3. Vector bundles over G/K

In this section we describe the geometry of $\mathrm{U}(p, q)/K$. Each element of G/K will determine a finite set of subspaces of $\mathbf{W}$. We discuss analysis on these subspaces of $\mathbf{W}$, specializing results of §1. The L^2-cohomology groups of these subspaces will be used to define holomorphic, homogeneous vector bundles over G/K. We explicitly describe the holomorphic structure of these vector bundles in coordinates.

3.1. **Definition.** Let $G = \mathrm{U}(p, q)$. By K we mean the maximal compact subgroup of G given by $K = \mathrm{U}(p, q) \cap \mathrm{U}(n) \cong \mathrm{U}(p) \times \mathrm{U}(q)$. For any $r \times s$ matrix A, we let $I(A) := I_s - {}^*AA$. We define the generalized unit disk $\mathscr{D}_{r,s}$ to be the bounded complex domain

$$\mathscr{D}_{r,s} := \{ \zeta \in \mathbb{C}^{r \times s} \mid I(\zeta) \gg 0 \}.$$

We will be working with the homogeneous space G/K.

3.2. **Lemma.** *The homogeneous space G/K is homeomorphic to $\mathscr{D}_{p,q}$ or to $\mathscr{D}_{q,p}$, and the natural action of $G = \mathrm{U}(p, q)$ on G/K corresponds to actions of G on $\mathscr{D}_{p,q}$ and $\mathscr{D}_{q,p}$ by fractional linear transformations. In particular, if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p, q)$, $\zeta \in \mathscr{D}_{p,q}$, and $\eta \in \mathscr{D}_{q,p}$, then*

$$g \cdot \zeta = (A\zeta + B)(C\zeta + D)^{-1}$$

and

$$g \cdot \eta = (C + D\eta)(A + B\eta)^{-1}.$$

These actions are equivalent under the bijection $\mathscr{D}_{p,q} \to \mathscr{D}_{q,p} : \zeta \mapsto {}^*\zeta$.

Proof. These results are well known, but a geometric proof will be instructive for the sequel. Let M be the set of all p-dimensional subspaces V of $\mathbb{C}^{p,q}$ which are positive (i.e., $\langle\langle z, z \rangle\rangle > 0$ for all nonzero $z \in V$), and let N be the set of all negative q-dimensional subspaces of $\mathbb{C}^{p,q}$. We will show that G/K is homeomorphic to N and to M, that $\mathscr{D}_{p,q}$ parametrizes N, and that $\mathscr{D}_{q,p}$ parametrizes M.

By Witt's Theorem the natural actions of G on M and on N are transitive. Recall from Definition 1.3 that we have fixed a basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^{p,q}$ so that the matrix of our hermitian form is $I_{p,q}$. We define subspaces $V_0 := \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_p$ and $V_1 := \mathbb{C}e_{p+1} \oplus \cdots \oplus \mathbb{C}e_n$ of $\mathbb{C}^{p,q}$. Clearly $V_0 \in M$ is positive and has isotropy group K, and analogously for $V_1 \in N$. Hence $G/K \cong M$ and $G/K \cong N$.

The plane V_0 above is the column span of the matrix $\begin{pmatrix} I_p \\ 0 \end{pmatrix}$. By our choice of basis of $\mathbb{C}^{p,q}$, any positive p-plane V is the column span of an $n \times p$ matrix whose top $p \times p$ minor is nonzero, so a matrix of the form $\begin{pmatrix} I_p \\ \eta \end{pmatrix}$ with $\eta \in \mathbb{C}^{q \times p}$. The condition on η for V to be positive is that

$$(I_p \quad {}^*\eta) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} I_p \\ \eta \end{pmatrix} = I_p - {}^*\eta\eta = I(\eta) \gg 0,$$

that is, $\eta \in \mathscr{D}_{q,p}$. Analogously, any negative q-plane in $\mathbb{C}^{p,q}$ is the column span of a matrix $\begin{pmatrix} \zeta \\ I_q \end{pmatrix}$ with $\zeta \in \mathscr{D}_{p,q}$. Thus $\mathscr{D}_{q,p}$ parametrizes M and $\mathscr{D}_{p,q}$ parametrizes N.

Now let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(p, q)$, and let $\zeta \in \mathscr{D}_{p,q}$ and $\eta \in \mathscr{D}_{q,p}$. The stated actions of g by fractional linear transformations result from the natural actions of $\mathrm{U}(p, q)$ on N,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \zeta \\ I_q \end{pmatrix} = \begin{pmatrix} (A\zeta + B)(C\zeta + D)^{-1} \\ I_q \end{pmatrix} (C\zeta + D),$$

and on M,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_p \\ \eta \end{pmatrix} = \begin{pmatrix} I_p \\ (C + D\eta)(A + B\eta)^{-1} \end{pmatrix} (A + B\eta).$$

That these actions are equivalent under $\zeta \mapsto {}^*\zeta$ follows from the defining properties (1.4) of $g \in \mathrm{U}(p, q)$. □

The generalized unit disks $\mathscr{D}_{p,q}$ and $\mathscr{D}_{q,p}$ are bounded realizations of G/K. Each element of G/K will now determine various subspaces of $\mathbf{W}$.

3.3. **Definition.** For any $\zeta \in \mathscr{D}_{p,q}$ and any integer s such that $0 \leq s \leq k$, we define a subspace $V_s(\zeta)$ of $\mathbf{W}$ by

$$V_s(\zeta) := \left\{ \begin{pmatrix} u_{11} & \zeta u_{22} \\ {}^*\zeta u_{11} & u_{22} \end{pmatrix} \in \mathbf{W} \mid u_{11} \in \mathbb{C}^{p \times (k-s)},\ u_{22} \in \mathbb{C}^{q \times s} \right\}.$$

If we define the matrices

$$R(\zeta) := \begin{pmatrix} I_p & \zeta \\ {}^*\zeta & I_q \end{pmatrix} \qquad \text{and} \qquad u := \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix},$$

then $V_s(\zeta)$ is the set of all $R(\zeta)u$ with $u \in V_s(0) \cong \mathbb{C}^{p(k-s)+qs}$.

The integer s will eventually be determined by the choice of a representation space $\mathscr{H}(\lambda)$ to be the number $s(\lambda)$ of negative entries in the highest weight of $\lambda \in \widehat{\mathrm{U}(k)} \subset \mathbb{Z}^k$, as defined in the previous section.

3.4. **Example.** Consider the case $k = 1$, so that $\mathbf{W} \cong \mathbb{C}^{p,q}$. Then $s = 0$ or $s = 1$. If $s = 0$, then $V_0(\zeta)$ is the set of all vectors $\begin{pmatrix} I_p \\ {}^*\zeta \end{pmatrix} u$ with $u \in \mathbb{C}^p$, i.e., a positive p-plane in $\mathbb{C}^{p,q}$. If $s = 1$, then $V_1(\zeta)$ is the set of all vectors $\begin{pmatrix} \zeta \\ I_q \end{pmatrix} u$ with $u \in \mathbb{C}^q$, a negative q-plane in $\mathbb{C}^{p,q}$. The planes V_0 and V_1 in the proof of Lemma 3.2 are just $V_0(0)$ and $V_1(0)$ in this notation.

If we think of $\mathbf{W}$ as a direct sum of k copies of $\mathbb{C}^{p,q}$, one per column, then $V_s(\zeta)$ may be regarded as a direct sum of $k - s$ positive p-planes and s negative q-planes, one for each copy of $\mathbb{C}^{p,q}$.

We now wish to determine the L^2-cohomology spaces of $V_s(\zeta)$, as defined in §1. First, some more notation is necessary.

3.5. **Definition.** Writing V_s for $V_s(0) \cong \mathbb{C}^{p \times (k-s)} \oplus \mathbb{C}^{q \times s}$, we think of $u \in V_s$ as a coordinate on $V_s(\zeta)$ via the mapping $u \mapsto R(\zeta)u : V_s \to V_s(\zeta) \subset \mathbf{W}$. For $u, v \in V_s$, we denote by $\langle\langle u, v \rangle\rangle_\zeta$ the restriction to $V_s(\zeta)$ of our fixed hermitian form on $\mathbf{W}$, given by

$$\langle\langle u, v \rangle\rangle_\zeta := \langle\langle R(\zeta)u, R(\zeta)v \rangle\rangle = \langle I_{p,q} \mathscr{T}(\zeta)u, v \rangle$$

where

$$\mathcal{I}(\zeta) = \begin{pmatrix} I(^*\zeta) & 0 \\ 0 & I(\zeta) \end{pmatrix} \gg 0.$$

The signature of this form is $(p(k-s), qs)$. We will use

$$\langle u, v\rangle_\zeta := \langle \mathcal{I}(\zeta)u, v\rangle$$

as the fixed choice of positive definite hermitian form on $V_s(\zeta)$. We normalize Lebesgue measure $dm_\zeta(u)$ on $V_s(\zeta)$ so that

$$\int_{V_s} e^{-\langle \mathcal{I}(\zeta)u, u\rangle}\, dm_\zeta(u) = 1, \tag{3.6}$$

i.e., by Lemma 1.2(3),

$$dm_\zeta(u) = \det I(^*\zeta)^{k-s} \det I(\zeta)^s\, dm(u) = \det I(\zeta)^k\, dm(u).$$

Blattner and Rawnsley's Theorem [BR] describes the L^2-cohomology of $V_s(\zeta)$. Combining their result, Theorem 1.9, with Definition 2.2, we see that for any $\lambda \in \mathbb{Z}^k$,

$$\mathcal{H}_\lambda^{0,qs}(V_s(\zeta)) = \Big\{ f(u)e^{-\langle \mathcal{I}(\zeta)u, u\rangle/2} d\bar{u}_{22} \mid f \text{ is holomorphic in } u_{11}, \bar{u}_{22} \text{ and } f(ue^{i(\theta_1, \dots, \theta_k)}) = e^{i(\lambda_1\theta_1 + \cdots + \lambda_k\theta_k)} f(u) \Big\}. \tag{3.7}$$

In particular, f must have homogeneity degree λ_j in the jth column of u, hence must be a polynomial, and so these spaces are finite dimensional for any $\lambda \in \mathbb{Z}^k$. The holomorphicity condition on f further implies that the space $\mathcal{H}_\lambda^{0,qs}(V_s(\zeta))$ will be nonzero only when $\lambda \in \mathbb{Z}^k$ satisfies $\lambda_j \geq 0$ for $j \leq k-s$ and $\lambda_j \leq 0$ for $j > k-s$.

We wish to construct families of holomorphic, G-homogeneous vector bundles over G/K having the spaces $\mathcal{H}_\lambda^{0,qs}(V_s(\zeta))$ as fibers. The definition of G-homogeneity requires that each $g \in G$ should determine a bijective linear map from $\mathcal{H}_\lambda^{0,qs}(V_s(\zeta))$ onto $\mathcal{H}_\lambda^{0,qs}(V_s(g\cdot\zeta))$, for all $\zeta \in \mathcal{D}_{p,q}$. The form of this mapping is easily derived as a corollary of the proof of Lemma 3.2. First, some additional notation is necessary.

3.8. **Definition.** Let $\zeta \in \mathcal{D}_{p,q}$ and $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$. We let $j(\zeta, g) \in \mathrm{GL}(n, \mathbb{C})$ denote the matrix

$$j(\zeta, g) := \begin{pmatrix} A + B^*\zeta & 0 \\ 0 & C\zeta + D \end{pmatrix}.$$

3.9. **Corollary.** *The natural action of G on G/K induces the mapping*

$$\mathcal{H}_\lambda^{0,qs}(V_s(\zeta)) \to \mathcal{H}_\lambda^{0,qs}(V_s(g\cdot\zeta)) : \omega \mapsto \omega \circ j(\zeta, g)^{-1}.$$

The defining properties (1.4) of $g \in \mathrm{U}(p, q)$ imply that the matrix $\mathcal{I}(\zeta)$ transforms nicely with respect to $j(\zeta, g)$.

3.10. **Lemma.** *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ *and let* $\zeta \in \mathscr{D}_{p,q}$. *Then*

$$^*j(\zeta, g)\mathscr{I}(g \cdot \zeta)\, j(\zeta, g) = \mathscr{I}(\zeta).$$

We may now define our vector bundles.

3.11. **Definition.** Let $\lambda \in \widehat{\mathrm{U}(k)} \subset \mathbb{Z}^k$ and let $s = s(\lambda)$ be the number of strictly negative components of λ. We denote by $\mathbf{E}_\lambda \to G/K$ the vector bundle with fiber

$$\mathbf{E}_\lambda(\zeta) = \mathscr{H}_\lambda^{0,qs}(V_s(\zeta)) \otimes \mathbb{C}\, du_{11}$$

over any $\zeta \in \mathscr{D}_{p,q} \cong G/K$. Let $\mathscr{P}(V_s; \lambda)$ denote the polynomial space

$$\mathscr{P}(V_s; \lambda) := \{ f : V_s \to \mathbb{C} | f \text{ is a polynomial in } u_{11}, \bar{u}_{22} \text{ such that } f(ue^{i(\theta_1, \dots, \theta_k)}) = e^{i(\lambda_1\theta_1 + \cdots + \lambda_k\theta_k)} f(u) \}.$$

We define the map $J_\lambda : \mathscr{D}_{p,q} \times \mathscr{P}(V_s; \lambda) \to \mathbf{E}_\lambda$ by

$$J_\lambda(\zeta, f)(u) := \det I(\zeta)^k f(\mathscr{I}(\zeta)u)\, e^{-\langle \mathscr{I}(\zeta)u, u\rangle/2} d\bar{u}_{22} \wedge du_{11}.$$

3.12. **Proposition.** *The trivialization* J_λ *above endows the vector bundle* $\mathbf{E}_\lambda \to G/K$ *with the structure of a holomorphic vector bundle which is homogeneous for* $G = \mathrm{U}(p, q)$.

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, $\zeta \in \mathscr{D}_{p,q}$, and let $f \in \mathscr{P}(V_s; \lambda)$. From Corollary 3.9, the natural action of g on $\mathbf{E}_\lambda$ satisfies

$$(g \cdot J_\lambda(\zeta, f))(u) = J_\lambda(\zeta, f)(j(\zeta, g)^{-1}u).$$

After using the definition of J_λ, we simplify the right-hand side with Lemma 3.10. We conclude that

$$(g \cdot J_\lambda(\zeta, f))(u) = J_\lambda(g \cdot \zeta, f')(u)$$

where f' satisfies

$$f'(u) = \det(C\zeta + D)^s \det(^*A + \zeta^*B)^{k-s} f(^*j(\zeta, g)u)$$

and so depends holomorphically on $\zeta \in \mathscr{D}_{p,q}$. □

Finally, we record here for future reference the specialization of Lemma 1.10 to the vector space $V_s(\zeta)$. The orthogonal projection $P_{s,\zeta} : \mathscr{L}_2^{0,qs}(V_s(\zeta)) \to \mathscr{H}^{0,qs}(V_s(\zeta))$ is given by $P_{s,\zeta}(\varphi d\bar{u}_J) = 0$ unless $u_J = u_{22}$ and

$$P_{s,\zeta}(\varphi d\bar{u}_{22})(u) = d\bar{u}_{22} \int_{V_s} \varphi(v) K_{s,\zeta}(u, v)\, dm_\zeta(v), \tag{3.13}$$

where $K_{s,\zeta}(u, v) = e^{-\langle \mathscr{I}(\zeta)u, u\rangle/2} e^{\langle I(^*\zeta)u_{11}, v_{11}\rangle + \langle I(\zeta)v_{22}, u_{22}\rangle} e^{-\langle \mathscr{I}(\zeta)v, v\rangle/2}$.

4. Construction of the integral transform

Throughout this section we fix an element $\lambda \in \Lambda \subseteq \widehat{\mathrm{U}(k)}$. We identify λ with its highest weight in $\mathbb{Z}^k$, and we fix $s = s(\lambda)$ to be the number of strictly negative entries in λ as in 2.3. We let $\mathscr{H}(\lambda) \subset \mathscr{H}^{0,qk}(\mathbf{W})$ be a representation space for the highest weight representation σ_λ of G, as in 2.8, and we let the vector bundle $\mathbf{E}_\lambda \to G/K$ be defined as in 3.11. In this section we construct an integral transform

$$\Phi_\lambda : \mathscr{H}(\lambda) \to H^{0,0}(G/K, \mathbf{E}_\lambda).$$

The transform will be defined essentially as restriction of a differential form in $\mathscr{H}(\lambda)$ to a subspace $V_s(\zeta)$ of $\mathbf{W}$, followed by projection onto its cohomology class in $\mathscr{H}_\lambda^{0,qs}(V_s(\zeta))$. This transform Φ_λ intertwines the actions of G and is injective, so provides a geometric construction of the representation σ_λ in a space of sections of a vector bundle $\mathbf{E}_\lambda$ over G/K.

First, for any $\omega \in \mathscr{H}(\lambda)$, we must define what we mean by restriction of a $(0, qk)$-form to a qs-dimensional subspace of $\mathbf{W}$. Our restriction will involve the coefficient function only.

4.1. **Proposition.** *For any* $\omega = \varphi\, d\bar{z}_2 \in \mathscr{H}^{0,qk}(\mathbf{W})$ *and any* $\zeta \in \mathscr{D}_{p,q}$,

$$(\varphi | V_s(\zeta))\, d\bar{u}_{22} \in \mathscr{L}_2^{0,qs}(V_s(\zeta)).$$

Proof. We follow the method of proof of Lemma 4.3 in [M2]. First we let $G = KA^+K$ be the Cartan decomposition of G. For any $\zeta \in \mathscr{D}_{p,q} \cong G/K$, there exists an element $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K$ such that $\zeta' = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \cdot \zeta = A\zeta D^{-1} \in A^+$ is real and diagonal, with nonnegative diagonal entries. Note that

$$(\varphi | V_s(\zeta))\,(u) = l\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{-1} \left(l\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \varphi | V_s(\zeta') \right)(u).$$

Since left translation by any nonsingular matrix preserves square-integrability, $\varphi | V_s(\zeta)$ is square integrable if and only if $\varphi | V_s(\zeta')$ is square integrable. Thus it suffices to consider only diagonal, nonnegative $\zeta \in \mathscr{D}_{p,q}$.

The isomorphism [B, Theorem 2.1] of the Bargmann-Segal-Fock space with $L^2(\mathbb{R}^{nk})$ states that there exists $\phi \in L^2(\mathbb{R}^{nk})$ such that

$$\begin{aligned} \varphi(z) = {} & \exp(-\langle z_1, \Re(z_1)\rangle - \langle \Re(z_2), z_2\rangle) \\ & \times \int_{\mathbb{R}^{nk}} \exp(-\tfrac{1}{4}|t|^2) \exp(\langle z_1, t_1\rangle + \langle t_2, z_2\rangle)\, \phi(t)\, dt. \end{aligned}$$

We replace z by $R(\zeta)u = R(\zeta)(x + iy)$ where x, $y \in V_s$ have real entries, and we compute that

$$\begin{aligned}|\varphi(R(\zeta)(x+iy))| \\ &= \exp(-\langle (I_p+\zeta{}^t\zeta)x_{11}, x_{11}\rangle - \langle (I_q+{}^t\zeta\zeta)x_{22}, x_{22}\rangle) \\ &\quad\times \left| \int_{\mathbb{R}^{nk}} \exp(\langle x_{11}, t_{11}+\zeta t_{21}\rangle + \langle x_{22}, {}^t\zeta t_{12}+t_{22}\rangle) \right. \\ &\qquad \left. \times \exp(i(\langle y_{11}, t_{11}-\zeta t_{21}\rangle + \langle y_{22}, {}^t\zeta t_{12}-t_{22}\rangle)) \exp(-\tfrac{1}{4}|t|^2)\phi(t)\,dt \right| \\ &= \exp(-\langle (I_p+\zeta{}^t\zeta)x_{11}, x_{11}\rangle - \langle (I_q+{}^t\zeta\zeta)x_{22}, x_{22}\rangle) \\ &\quad\times \left| \int_{\mathbb{R}^{nk}} \exp(\langle x_{11}, t_{11}+2\zeta t_{21}\rangle + \langle x_{22}, 2{}^t\zeta t_{12}+t_{22}\rangle) \right. \\ &\qquad \left. \times \exp(i(\langle y_{11}, t_{11}\rangle - \langle y_{22}, t_{22}\rangle)) \exp(-\tfrac{1}{4}|\tilde{t}|^2)\phi(\tilde{t})\,dt \right|.\end{aligned}$$

In the integral above we replaced t_{11} and t_{22} by, respectively, $t_{11}+\zeta t_{21}$ and $t_{22}+{}^t\zeta t_{12}$, and we have written

$$\tilde{t} := \begin{pmatrix} t_{11}+\zeta t_{21} & t_{12} \\ t_{21} & t_{22}+{}^t\zeta t_{12} \end{pmatrix}.$$

Upon integrating to evaluate the norm of this function, we compute that

$$\begin{aligned}\int_{V_s} |\varphi(R(\zeta)(x+iy))|^2 dx\,dy \\ &= \int_{\mathbb{R}^{p(k-s)+qs}} \exp(-2\langle (I_p+\zeta{}^t\zeta)x_{11}, x_{11}\rangle - 2\langle (I_q+{}^t\zeta\zeta)x_{22}, x_{22}\rangle) \\ &\quad\times \int_{\mathbb{R}^{p(k-s)+qs}} \left| \int_{\mathbb{R}^{p(k-s)}\times\mathbb{R}^{qs}} \exp(i\langle y_{11}, t_{11}\rangle - i\langle y_{22}, t_{22}\rangle) \right. \\ &\qquad\times \left(\int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} \exp(\langle x_{11}, t_{11}+2\zeta t_{21}\rangle + \langle x_{22}, t_{22}+2{}^t\zeta t_{12}\rangle) \right. \\ &\qquad\qquad \left.\left. \times \exp(-\tfrac{1}{4}|\tilde{t}|^2)\phi(\tilde{t})\,dt_{12}dt_{21} \right) dt_{11}dt_{22} \right|^2 dy\,dx \\ &= \int_{\mathbb{R}^{p(k-s)+qs}} \exp(-2\langle (I_p+\zeta{}^t\zeta)x_{11}, x_{11}\rangle - 2\langle (I_q+{}^t\zeta\zeta)x_{22}, x_{22}\rangle) \\ &\quad\times \int_{\mathbb{R}^{p(k-s)}\times\mathbb{R}^{qs}} \left| \int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} \exp(\langle x_{11}, t_{11}+2\zeta t_{21}\rangle + \langle x_{22}, t_{22}+2{}^t\zeta t_{12}\rangle) \right. \\ &\qquad\qquad \left. \times \exp(-\tfrac{1}{4}|\tilde{t}|^2)\phi(\tilde{t})\,dt_{12}dt_{21} \right|^2 dt_{11}dt_{22}\,dx\end{aligned}$$

by the Plancherel Theorem. We now replace t_{11} and t_{22} by, respectively, $t_{11}-\zeta t_{21}$ and $t_{22}-{}^t\zeta t_{12}$ and use the Cauchy-Schwarz inequality, computing that

$$\begin{aligned}\int_{V_s} |\varphi(R(\zeta)(x+iy))|^2 dx\,dy \\ &= \int_{\mathbb{R}^{p(k-s)+qs}} \exp(-2\langle (I_p+\zeta{}^t\zeta)x_{11}, x_{11}\rangle - 2\langle (I_q+{}^t\zeta\zeta)x_{22}, x_{22}\rangle)\end{aligned}$$

$$\times \int_{\mathbb{R}^{p(k-s)}\times\mathbb{R}^{qs}} \left| \int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} \exp(\langle x_{11}, t_{11} + \zeta t_{21}\rangle + \langle x_{22}, t_{22} + {}^t\zeta t_{12}\rangle) \right.$$
$$\left. \times \exp(-\tfrac{1}{4}|t|^2)\phi(t)\, dt_{12}dt_{21} \right|^2 dt_{11}dt_{22}\, dx$$
$$\leq \int_{\mathbb{R}^{p(k-s)+qs}} \exp(-2\langle (I_p + \zeta{}^t\zeta)x_{11}, x_{11}\rangle - 2\langle (I_q + {}^t\zeta\zeta)x_{22}, x_{22}\rangle)$$
$$\times \int_{\mathbb{R}^{p(k-s)}\times\mathbb{R}^{qs}} \exp(-\tfrac{1}{2}(|t_{11}|^2 + |t_{22}|^2)) \Big(\int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} \exp(-\tfrac{1}{2}(|\xi|^2 + |\eta|^2))$$
$$\times \exp(2\langle x_{11}, t_{11} + \zeta\xi\rangle + 2\langle x_{22}, t_{22} + {}^t\zeta\eta\rangle)\, d\xi\, d\eta \Big)$$
$$\times \Big(\int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} |\phi(t)|^2\, dt_{12}dt_{21} \Big)\, dt_{11}dt_{22}\, dx$$
$$= C \int_{\mathbb{R}^{p(k-s)+qs}} \exp(-2\langle (I_p + \zeta{}^t\zeta)x_{11}, x_{11}\rangle - 2\langle (I_q + {}^t\zeta\zeta)x_{22}, x_{22}\rangle)$$
$$\times \int_{\mathbb{R}^{p(k-s)}\times\mathbb{R}^{qs}} \exp(-\tfrac{1}{2}(|t_{11}|^2 + |t_{22}|^2))$$
$$\times \exp(2\langle x_{11}, t_{11}\rangle + 2\langle x_{22}, t_{22}\rangle) \exp(2\langle \zeta{}^t\zeta x_{11}, x_{11}\rangle + 2\langle {}^t\zeta\zeta x_{22}, x_{22}\rangle)$$
$$\times \int_{\mathbb{R}^{q(k-s)}\times\mathbb{R}^{ps}} |\phi(t)|^2\, dt_{12}dt_{21}\, dt_{11}dt_{22}\, dx$$
$$= C \int_{\mathbb{R}^{nk}} \Big(\int_{\mathbb{R}^{p(k-s)}} \exp(-2\langle x_{11} - \tfrac{1}{2}t_{11}, x_{11} - \tfrac{1}{2}t_{11}\rangle)\, dx_{11} \Big)$$
$$\times \Big(\int_{\mathbb{R}^{qs}} \exp(-2\langle x_{22} - \tfrac{1}{2}t_{22}, x_{22} - \tfrac{1}{2}t_{22}\rangle)\, dx_{22} \Big) |\phi(t)|^2\, dt$$
$$= C' \int_{\mathbb{R}^{nk}} |\phi(t)|^2\, dt < \infty.$$

Here we have used Lemma 1.2 to evaluate integrals in ξ, η, x_{11}, and x_{22}, and have labelled the resulting constants as C and C'. □

4.2. **Definition.** Let $H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda)$ be the space of holomorphic sections of $\mathbf{E}_\lambda$ over $G/K \cong \mathscr{D}_{p,q}$. We define a mapping

$$\Phi_\lambda : \mathscr{H}(\lambda) \to H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda)$$

so that for $\omega = \varphi\, d\bar{z}_2 \in \mathscr{H}(\lambda)$,

$$(\Phi_\lambda(\omega)(\zeta))(u) := P_{s,\zeta}(\varphi | V_s(\zeta) d\bar{u}_{22}) \wedge du_{11}.$$

By (3.13) and Definition 3.3,

$$(\Phi_\lambda(\omega)(\zeta))(u) = d\bar{u}_{22} \wedge du_{11} \int_{V_s} \varphi(R(\zeta)v)\, K_{s,\zeta}(u, v)\, dm_\zeta(v), \tag{4.3}$$

with

$$K_{s,\zeta}(u, v) = \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle + \langle I({}^*\zeta)u_{11}, v_{11}\rangle$$
$$+ \langle I(\zeta)v_{22}, u_{22}\rangle - \tfrac{1}{2}\langle \mathscr{I}(\zeta)v, v\rangle).$$

4.4. **Lemma.** *The transform* Φ_λ *is well defined.*

Proof. If we choose any differential form $\omega = \varphi d\bar{z}_2 \in \mathscr{H}^{0,qk}(\mathbf{W})$, Proposition 4.1 implies that $\varphi|V_s(\zeta)\,d\bar{u}_{22} \in \mathscr{L}_2^{0,qs}(V_s(\zeta))$, so the integral $P_{s,\zeta}(\varphi|V_s(\zeta)\,d\bar{u}_{22})$ converges. If $\omega \in \mathscr{H}(\lambda)$, then Definition 2.2 and (2.9) imply $P_{s,\zeta}(\varphi|V_s(\zeta)\,d\bar{u}_{22}) \in \mathscr{H}_\lambda^{0,qs}(V_s(\zeta))$, so that $\Phi_\lambda(\omega)(\zeta) \in \mathbf{E}_\lambda(\zeta)$. Finally, we must verify that $\Phi_\lambda(\omega)$ is a holomorphic section of $\mathbf{E}_\lambda$. Given $\omega = \varphi d\bar{z}_2 \in \mathscr{H}(\lambda)$, let $\varphi(z) = f(z)\exp(-\frac{1}{2}|z|^2)$, and define

$$\psi_\zeta(u) := \int_{V_s} f(R(\zeta)v)\exp(-\langle v, v\rangle)\exp(\langle u_{11}, v_{11}\rangle + \langle v_{22}, u_{22}\rangle)\,dm(v). \tag{4.5}$$

It follows from (4.3) and (3.6) and Definitions 3.3, 3.5, and 3.11 that

$$\begin{aligned}(\Phi_\lambda(\omega)(\zeta))(u) &= \det \mathscr{J}(\zeta)^k \psi_\zeta(\mathscr{J}(\zeta)u)\exp(-\tfrac{1}{2}\langle \mathscr{J}(\zeta)u, u\rangle)d\bar{u}_{22}\wedge du_{11} \\ &= J_\lambda(\zeta, \psi_\zeta)(u).\end{aligned}$$

The function ψ_ζ depends holomorphically on $\zeta \in \mathscr{D}_{p,q}$ since

$$f(R(\zeta)v) = f\begin{pmatrix} v_{11} & \zeta v_{22} \\ {}^t\bar{\zeta} v_{11} & v_{22}\end{pmatrix}$$

depends holomorphically on entries in the top $p \times k$ block and antiholomorphically on entries in the bottom $q \times k$ block. Thus the transform Φ_λ is well defined. □

We now verify the $\mathrm{SU}(p, q)$-equivariance of Φ_λ.

4.6. **Theorem.** *For all* $\omega \in \mathscr{H}(\lambda)$ *and all* $g \in \mathrm{U}(p, q)$, *the mapping*

$$\Phi_\lambda : \mathscr{H}(\lambda) \to H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda)$$

satisfies

$$(\det g)^{k-s} g\cdot \Phi_\lambda(\omega) = \Phi_\lambda(g\cdot\omega).$$

Furthermore, Φ_λ *is injective.*

Proof. For the proof of the first statement, we examine the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{H}(\lambda) & \xrightarrow{\sigma_\lambda(g)} & \mathscr{H}(\lambda) \\ \Phi_\lambda \downarrow & & \downarrow \Phi_\lambda \\ H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda) & \xrightarrow[g]{} & H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda) \end{array}$$

where the action of $g \in G$ on $H^{0,0}(\mathscr{D}_{p,q}, \mathbf{E}_\lambda)$ is given by Lemma 3.2 and Corollary 3.9 to be

$$\begin{aligned}((g\cdot s)(\zeta))(u) &= (s(g^{-1}\cdot\zeta))(j(\zeta, g^{-1})u) \\ &= s(g^{-1}\cdot\zeta)\begin{pmatrix} (A+B^*\zeta)u_{11} & 0 \\ 0 & (C\zeta + D)u_{22}\end{pmatrix}.\end{aligned} \tag{4.7}$$

Here we let $g^{-1} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$.

First we describe the mapping $g \circ \Phi_\lambda$. Let $\omega = \varphi d\bar{z}_2 \in \mathscr{H}(\lambda)$. We recall from (4.3) that

$$(\Phi_\lambda(\omega)(\zeta))(u) = d\bar{u}_{22} \wedge du_{11} \int_{V_s} \varphi(R(\zeta)v)\, K_{s,\zeta}(u, v)\, dm_\zeta(v).$$

Thus, (4.7) implies that

$$\begin{aligned}((g \cdot \Phi_\lambda(\omega))(\zeta))(u) \\ &= \det{}^*(C\zeta + D)^s \det(A + B^*\zeta)^{k-s} d\bar{u}_{22} \wedge du_{11} \\ &\times \int_{V_s} \varphi(R(g^{-1} \cdot \zeta)v) K_{s, g^{-1}\cdot\zeta}(j(\zeta, g^{-1})u, v)\, dm_{g^{-1}\cdot\zeta}(v).\end{aligned}$$

We replace v by $j(\zeta, g^{-1})v$ in the integral and we simplify using Lemma 3.10. We conclude that

$$\begin{aligned}((g \cdot \Phi_\lambda(\omega))(\zeta))(u) \\ &= \det{}^*(C\zeta + D)^s \det(A + B^*\zeta)^{k-s} d\bar{u}_{22} \wedge du_{11} \\ &\times \int_{V_s} \varphi(R(g^{-1} \cdot \zeta) j(\zeta, g^{-1})v) K_{s,\zeta}(u, v)\, dm_\zeta(v),\end{aligned} \tag{4.8}$$

where the argument of φ is

$$R(g^{-1} \cdot \zeta) j(\zeta, g^{-1})v = \begin{pmatrix} (A + B^*\zeta)v_{11} & (A\zeta + B)v_{22} \\ (C + D^*\zeta)v_{11} & (C\zeta + d)v_{22} \end{pmatrix}.$$

Now we describe the mapping $\Phi_\lambda \circ \sigma_\lambda(g)$. Let $\omega = \phi d\bar{z}_2 \in \mathscr{H}(\lambda)$ and let $g \in \mathrm{U}(p, q)$ be such that $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that $g = \begin{pmatrix} {}^*A & -{}^*C \\ -{}^*B & {}^*D \end{pmatrix}$. We recall from 1.11 that

$$(\sigma_\lambda(g)\omega)(z) = \det{}^* D^k d\bar{z}_2 \int_{\mathbf{W}} \varphi(g^{-1}w) K(z, w)\, dm(w).$$

Thus, by (4.3),

$$\begin{aligned}(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta))(u) \\ &= \det{}^* D^k\, d\bar{u}_{22} \wedge du_{11} \\ &\times \int_{\mathbf{W}} \varphi(g^{-1}w) \int_{V_s} K(R(\zeta)v, w)\, K_{s,\zeta}(u, v)\, dm_\zeta(v)\, dm(w).\end{aligned}$$

We use Lemma 1.2(3) to evaluate the inner integral in v, referring to the descriptions of the functions K and $K_{s,\zeta}$ in 1.10 and (4.3), respectively, and we conclude that

$$\begin{aligned}(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta))\,(u) \\ &= \det{}^* D^k \det I(\zeta)^k \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle)\, d\bar{u}_{22} \wedge du_{11} \\ &\times \int_{\mathbf{W}} \varphi(g^{-1}w) \exp(-\tfrac{1}{2}\langle w, w\rangle) \\ &\qquad \times \exp(\langle \zeta w_{21} + I({}^*\zeta)u_{11}, w_{11}\rangle + \langle w_{22}, {}^*\zeta w_{12} + I(\zeta)u_{22}\rangle)\, dm(w).\end{aligned}$$

A comparison of the integral above to (4.8) indicates that we wish to evaluate the integrals in w_{12} and w_{21} over $\mathbb{C}^{p\times s}$ and $\mathbb{C}^{q\times(k-s)}$, respectively. We use the description of φ from Theorem 1.9 to rewrite the integral above as

$$\begin{aligned}
&\big(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta)\big)\,(u)\\
&\quad= \det {}^*D^k \det I(\zeta)^k \exp(-\tfrac{1}{2}\langle\mathscr{I}(\zeta)u\,,\,u\rangle)\,d\bar{u}_{22}\wedge du_{11}\\
&\qquad\times\int_{\mathbf{W}} f(g^{-1}w)\exp(-\langle\tfrac{1}{2}({}^*g^{-1}g^{-1}+I_n)w\,,\,w\rangle)\\
&\qquad\qquad\times\exp(\langle\zeta w_{21}+I({}^*\zeta)u_{11}\,,\,w_{11}\rangle+\langle w_{22}\,,\,{}^*\zeta w_{12}+I(\zeta)u_{22}\rangle)\,dm(w)\\
&\quad= \det {}^*D^k \det I(\zeta)^k \exp(-\tfrac{1}{2}\langle\mathscr{I}(\zeta)u\,,\,u\rangle)\,d\bar{u}_{22}\wedge du_{11}\\
&\qquad\times\int_{\mathbf{W}} f(g^{-1}w)\exp(-(|w_{21}|^2+|w_{12}|^2))\\
&\qquad\qquad\times\exp(-(|Aw_{11}+Bw_{21}|^2+|Cw_{12}+Dw_{22}|^2))\\
&\qquad\qquad\times\exp(\langle\zeta w_{21}+I({}^*\zeta)u_{11}\,,\,w_{11}\rangle+\langle w_{22}\,,\,{}^*\zeta w_{12}+I(\zeta)u_{22}\rangle)\,dm(w).
\end{aligned}$$

The argument of f in the integral above (cf. (4.8)) is

$$g^{-1}w=\begin{pmatrix} Aw_{11}+Bw_{21} & Aw_{12}+Bw_{22}\\ Cw_{11}+Dw_{21} & Cw_{12}+Dw_{22}\end{pmatrix}.$$

We now perform substitutions in the integral above which replace w_{11} by $v_{11}-A^{-1}B(w_{21}-{}^*\zeta v_{11})$ and w_{22} by $v_{22}-D^{-1}C(w_{12}-\zeta v_{22})$. These substitutions replace the expressions $Aw_{11}+Bw_{21}$ and $Cw_{12}+Dw_{22}$ by the expressions $(A+B^*\zeta)v_{11}$ and $(C\zeta+D)v_{22}$, respectively. We then calculate that

$$\begin{aligned}
&\big(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta)\big)\,(u)\\
&\quad= |\det(I_p+A^{-1}B^*\zeta)|^{2(k-s)}\,|\det(I_q+D^{-1}C\zeta)|^{2s}\det {}^*D^k\\
&\qquad\times\det I(\zeta)^k\exp(-\tfrac{1}{2}\langle\mathscr{I}(\zeta)u\,,\,u\rangle)\,d\bar{u}_{22}\wedge du_{11}\\
&\qquad\times\int_{\mathbf{W}} f\begin{pmatrix} (A+B^*\zeta)v_{11} & (A\zeta+B)v_{22}+{}^*A^{-1}(w_{12}-\zeta v_{22})\\ (C+D^*\zeta)v_{11}+{}^*D^{-1}(w_{21}-{}^*\zeta v_{11}) & (C\zeta+D)v_{22}\end{pmatrix}\\
&\qquad\times\exp(-(|(A+B^*\zeta)v_{11}|^2+|(C\zeta+D)v_{22}|^2+|w_{21}|^2+|w_{12}|^2))\\
&\qquad\times\exp(\langle\zeta w_{21}+I({}^*\zeta)u_{11}\,,\,v_{11}-A^{-1}B(w_{21}-{}^*\zeta v_{11})\rangle)\\
&\qquad\times\exp(\langle v_{22}-D^{-1}C(w_{12}-\zeta v_{22})\,,\,{}^*\zeta w_{12}+I(\zeta)u_{22}\rangle)\\
&\qquad\qquad\qquad\times dm(v)\,dm(w_{12})\,dm(w_{21}).
\end{aligned}$$

We now perform the substitutions which replace w_{12} by $\xi+\zeta v_{22}$ and w_{21} by $\eta+{}^*\zeta v_{11}$ and calculate that

$$
\begin{aligned}
&(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta))\,(u) \\
&\quad = |\det(I_p + A^{-1}B^*\zeta)|^{2(k-s)}\,|\det(I_q + D^{-1}C\zeta)|^{2s}\,\det {}^*D^k \\
&\qquad \times \det I(\zeta)^k \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle)\, d\bar{u}_{22}\wedge du_{11} \\
&\qquad \times \int_{V_s} \exp(-(|(A+B^*\zeta)v_{11}|^2 + |(C\zeta + D)v_{22}|^2)) \\
&\qquad \times \exp(\langle I({}^*\zeta)u_{11}, v_{11}\rangle + \langle v_{22}, I(\zeta)u_{22}\rangle) \\
&\qquad \times \int_{\mathbb{C}^{p\times s}\oplus\mathbb{C}^{q\times(k-s)}} f\begin{pmatrix} (A+B^*\zeta)v_{11} & (A\zeta + B)v_{22} + {}^*A^{-1}\xi \\ (C + D^*\zeta)v_{11} + {}^*D^{-1}\eta & (C\zeta + D)v_{22}\end{pmatrix} \\
&\qquad \times \exp(-\langle (I_q + {}^*B^*A^{-1}\zeta)(\eta + {}^*\zeta v_{11}), \eta\rangle - \langle I({}^*\zeta)u_{11}, A^{-1}B\eta\rangle) \\
&\qquad \times \exp(-\langle (I_p + \zeta D^{-1}C)\xi, \xi + \zeta v_{22}\rangle - \langle D^{-1}C\xi, I(\zeta)u_{22}\rangle) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times dm(\xi)\, dm(\eta)\, dm(v).
\end{aligned}
$$

We now use Lemma 1.2(3) to evaluate the inner integral with respect to ξ and η and compare the resulting expression to (4.8), allowing us to conclude that

$$
\begin{aligned}
&(\Phi_\lambda(\sigma_\lambda(g)\omega)(\zeta))\,(u) \\
&\quad = \frac{|\det(I_p + A^{-1}B^*\zeta)|^{2(k-s)}\,|\det(I_q + D^{-1}C\zeta)|^{2s}\,\det {}^*D^k}{\det(I_q + {}^*B^*A^{-1}\zeta)^{k-s}\,\det(I_p + \zeta D^{-1}C)^s} \\
&\qquad \times \det I(\zeta)^k \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle)\, d\bar{u}_{22}\wedge du_{11} \\
&\qquad \times \int_{V_s} f\begin{pmatrix} (A+B^*\zeta)v_{11} & (A\zeta + B)v_{22} \\ (C + D^*\zeta)v_{11} & (C\zeta + D)v_{22}\end{pmatrix} \\
&\qquad\qquad \times \exp(-(|(A+B^*\zeta)v_{11}|^2 + |(C\zeta + D)v_{22}|^2)) \\
&\qquad\qquad\qquad \times \exp(\langle I({}^*\zeta)u_{11}, v_{11}\rangle + \langle v_{22}, I(\zeta)u_{22}\rangle)\, dm(v) \\
&\quad = (\det {}^*D\, \det A^{-1})^{k-s} \det(A + B^*\zeta)^{k-s} \det {}^*(C\zeta + D)^s d\bar{u}_{22}\wedge du_{11} \\
&\qquad \times \int_{V_s} \varphi(R(g^{-1}\cdot\zeta) j(\zeta, g^{-1})v) K_{s,\zeta}(u, v)\, dm_\zeta(v) \\
&\quad = (\det g)^{k-s}\,((g\cdot\Phi_\lambda(\omega))(\zeta))\,(u).
\end{aligned}
$$

Here we have used the fact that $\det(I_l + XY) = \det(I_m + YX)$ for any $l\times m$ matrix X and $m\times l$ matrix Y and the fact that $\det {}^*D \det A^{-1} = \det\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1} = \det g$. This completes the proof of the first statement of the theorem.

It remains to prove that Φ_λ is injective. We know that the module $\mathscr{H}(\lambda)$ is irreducible for the action of $SU(p, q)$, and we have just shown that Φ_λ intertwines the $SU(p, q)$-actions. Thus Φ_λ is either injective on $\mathscr{H}(\lambda)$ or is identically zero. We show that $\Phi_\lambda(\omega_\lambda)$ is not zero, where ω_λ is the highest weight vector in $\mathscr{H}(\lambda)$ defined in (2.4). We compute that

$$\begin{aligned}
&(\Phi_\lambda(\omega_\lambda)(\zeta))(u)\\
&\quad = d\bar{u}_{22} \wedge du_{11} \int_{V_s} f_\lambda(R(\zeta)v) \exp(-\tfrac{1}{2}|R(\zeta)v|^2) K_{s,\zeta}(u, v)\, dm_\zeta(v)\\
&\quad = \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle) d\bar{u}_{22} \wedge du_{11} \int_{V_s} f_\lambda(v) \exp(-\langle v_{11} - I({}^*\zeta)u_{11}, v_{11}\rangle)\\
&\qquad\qquad \times \exp(-\langle v_{22}, v_{22} - I(\zeta)u_{22}\rangle)\, dm_\zeta(v)\\
&\quad = \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle) d\bar{u}_{22} \wedge du_{11}\\
&\qquad \times \int_{V_s} f_\lambda(v + \mathscr{I}(\zeta)u) \exp(-\langle v_{11}, v_{11} + I({}^*\zeta)u_{11}\rangle)\\
&\qquad\qquad \times \exp(-\langle v_{22} + I(\zeta)u_{22}, v_{22}\rangle)\, dm_\zeta(v),
\end{aligned}$$

where we have replaced v in the integral by $v + \mathscr{I}(\zeta)u$. Using Lemma 1.2(3) to evaluate the integral, we now compute that

$$(\Phi_\lambda(\omega_\lambda)(\zeta))(u) = \det I(\zeta)^k f_\lambda(\mathscr{I}(\zeta)u) \exp(-\tfrac{1}{2}\langle \mathscr{I}(\zeta)u, u\rangle)\, d\bar{u}_{22} \wedge du_{11}.$$

Since $I(\zeta)$ is positive definite, $\det I(\zeta) \neq 0$. It thus remains to verify that

$$f_\lambda(\mathscr{I}(\zeta)u) = f_\lambda \begin{pmatrix} I({}^*\zeta)u_{11} & 0 \\ 0 & I(\zeta)u_{22} \end{pmatrix}$$

is not identically zero in u and ζ. Choose $u_{11} = (\alpha_{ij}) \in \mathbb{C}^{p\times(k-s)}$ so that $\alpha_{ij} = 1$ if $i + j = p + 1$ and $\alpha_{ij} = 0$ otherwise. Then the columns of the matrix product $I({}^*\zeta)u_{11}$ are just the last $k - s$ columns of $I({}^*\zeta)$ in reverse order, if $k - s \leq p$. If $k - s > p$, then the columns of $I({}^*\zeta)u_{11}$ are the p columns of $I({}^*\zeta)$ in reverse order followed by columns of zeros. Note that in the definition of f_λ in 2.3 we consider only the first r columns of $I({}^*\zeta)u_{11}$, where $r = r(\lambda) \leq p$. Similarly, choose $u_{22} = (\beta_{ij}) \in \mathbb{C}^{q\times s}$ so that $\beta_{ij} = 1$ if $i + j = s + 1$ and $\beta_{ij} = 0$ otherwise. Since $s < q$, we know that the columns of the product $I(\zeta)u_{22}$ are the first s columns of $I(\zeta)$ in reverse order. It now follows from the positive definiteness of $I({}^*\zeta)$ and $I(\zeta)$ that each determinant factor in the definition of $f_\lambda(\mathscr{I}(\zeta)u)$ (see 2.3) is positive. This implies that $f_\lambda(\mathscr{I}(\zeta)u) > 0$. We conclude that $\Phi_\lambda(\omega_\lambda) \neq 0$, and so Φ_λ is injective on $\mathscr{H}(\lambda)$. This completes the proof of the theorem. □

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Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078-0613